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AN EXACT METHOD TO CALCULATE THE NUCLEAR BINDING ENERGY

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ABSTRACT

The present study covers a part of the history of the nuclear binding energy. It is based on the formula of *Albert Einstein* mass-energy equivalence ($E = mc^2$). We present in this paper a brief history of Aston's whole number, mass-defect and nuclear binding energy, its exact definition, and especially its sign that raises fierce controversy between physicists and students.

Keywords: Aston's whole number rule, mass-defect, nuclear binding energy.

INTRODUCTION

Where does the sign minus (–) which exists in front of binding energy of nuclei come from? And what is its signification? Why certain authors consider it as being negative, and others, positive? Students ask worrying questions. When reading the various discussions of nuclear binding energy by different authors, it is easy to get confused. Binding energy is always negative. When they talk about its magnitude, they mean its absolute value, so they just state the positive number. Thus, the physicists have been very sloppy in definitions (Rideout, 2011). To clarify this basic concept, we must find answers to the following question: How does Aston's packing fraction become the binding energy?

Aston (1919) introduced three new concepts to determine masses of individual atoms and their isotopes: 1) *Whole Number Rule*, 2) *Mass-defect*, and 3) *Packing fraction*.

Firstly, the *Aston's Whole Number Rule* stated that the nuclei masses are integer multiples of a certain elementary particle of mass into the nucleus. This rule was a preliminary model for the atomic nucleus but its limitation was that the only particles known at this epoch were the proton and the electron. It was therefore proposed that the nucleus of an isotope of mass M and charge Z, both being integers, consisted of M protons and M-Z electrons. Thus, for example, the nucleus of ${}^{7}_{3}\mathrm{Li}$ consisted of 7 protons and 4 electrons, while that of ${}^{7}_{3}\mathrm{Li}$ consisted of 6 protons and 3 electrons (Squires, 1998). Although this model gave the correct mass and electric charge of the nucleus, and appeared to satisfy the whole number rule, it does not function fully well because it presents some defects. The conservation of electric and magnetic properties of the whole atom was not verified. For example, atom should be an electrically neutral particle i.e., sum of charges equal to zero; also the spins of some of the nuclei were anomalous.

The discovery of the neutron by Chadwick (Chadwick, 1932) in 1932 removed these problems. The actual model is that a nucleus of atomic number Z and mass number A contains Z protons and N neutrons. Someway, the mass number of an atom is the Aston's whole number rule (Aston, 1920). Moreover, isotopes are thus nuclei with the same number of protons and a different number of neutrons.

Secondly, the *mass-defect* is the deviation of the atomic mass M_A from its whole number A. Its mathematical expression was:

$$\Delta M = M_A - A \qquad (1)$$

Where atomic mass M_A and whole number A are molar masses, i.e., in kg/mol. The mass of an individual atom is equals to the atomic mass (M_A) divided by Avogadro's constant (N_A). Mass defect may be defined as the amount of mass which would be converted into energy if a particular atom has to be assembled from its constituents (Fig. 1). The energy equivalent of mass defect is a measure of binding energy of the nucleus.

Thirdly, *Packing fraction*. Inside nuclei, the nucleons are very tightly packed together (Fig. 1). It can be shown that in the original process of the formation of these, energy must be released in very large amount before a stable packing state is reached. The loss of energy which is

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related to the binding forces between the nucleons means a corresponding loss of mass (mass-defect). The expression of Aston's *packing fraction* is

$$f = \frac{\Delta m}{A} = \frac{M_A - A}{A} \qquad (2)$$

Since M_A and A are expressed in molar atomic mass unit (kg/mol) i.e., the packing fraction is dimensionless.

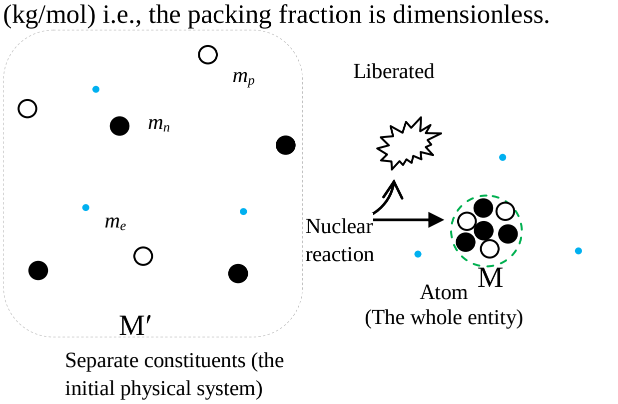

Fig. 1. Formation of the $^{6}\mathrm{Li}$ atom.

Thus, the actual molar mass of one of more stable isotopes of chlorine $^{37}_{17}\mathrm{Li}$ is $M_{cl}=36{,}965\ 902\ 574\ g/mol$. According to equation (2) the packing fraction therefore is

$$f = \frac{36{,}965\,902\,574 - 37}{37} = -0{,}000921 \qquad (3)$$

Since packing fraction number are very small, It is generally multiplied by 1×10^4 to be significant; i.e., $f = -9{,}21$ (Sharma *et al*., 2001). In 1927, Aston reported a first curve of the packing fraction, as shown in Figure 2.

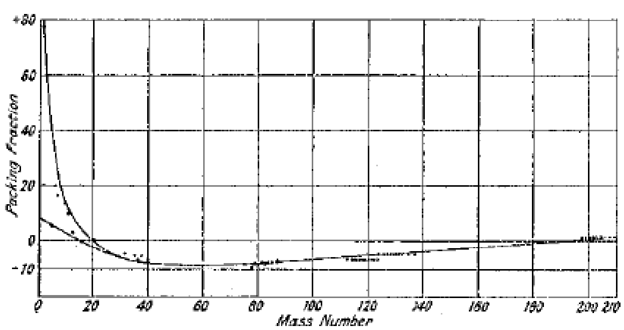

Fig. 2. The 1927 Aston's packing fraction curve (Audi, 2006; Raj, 2008).

RESULTS AND DISCUSSION

In 1919, F. W. Aston had invented the packing fraction to determine the atomic masses (Squires, 1998; Aston, 1942). Rapidly, packing fraction proved to be a very important physical quantity because it was not only used in the determination of the mass of atoms but it was considered as being a strong indicator of the stability of nuclei. Even in the relation (2), Aston's packing fraction is dimensionless and appears as if it means nothing. Aston's intuition with several other contemporary pioneer Chemists and Physicists was good: the packing fraction has been understood as being the nuclear binding energy or stability of the nuclei or the like. In order to show how Aston's packing fraction becomes a binding energy of nuclei, we need to use for the value of A into the numerator of (2) the sum of the masses of separate atom's constituents (protons, neutrons and electrons), rather than the mass number itself (Elsasser, 1933). Then, the amount A in the numerator of Aston's packing fraction will be quite different from the amount A in its denominator. Therefore, we shall use symbol *m(A,Z)* instead of symbol A into the numerator. Nevertheless, symbol A in the denominator is going to keep its original signification; it must represent the Aston's whole number rule. Then, we would expect that *m(A,Z)* would be given by the atomic number Z multiplied by the mass of the electron m_e plus the mass of the proton m_p (and this is the mass of hydrogen atom) plus the number of neutrons (N=A–Z) multiplied by the mass of the neutron m_n. Mathematically,

$$m(A,Z)=Z(m_e+m_p)+(A-Z)m_n \qquad (4)$$

For example, for the helium atom, $^{4}_{2}\mathrm{He}$, with two electrons, two protons and two neutrons, we would then anticipate an atomic mass of $m(A,Z)=2m_e+2m_p+2m_n$ according to relation (4). Generally, the masses of the atom's constituents are $m_p=1{,}007\ 275\ 47\ u$, for the proton, $m_n=1{,}008\ 664\ 92\ u$, for the neutron, and $m_e=0{,}000\ 548\ 58\ u$, for the electron. Then, $m(4,2)=2(0{,}000\ 548\ 58 + 1{,}007\ 275\ 47 + 1{,}008\ 664\ 92)=4{,}034\ 077\ 06\ u$.

On the other hand, the mass of an atom is $m(^{4}\mathrm{He})=4{,}002\ 603\ 25\ u$ according to the experimental measurements. The difference between the calculated and measured values which in the case of the helium equals $-0{,}031\ 473\ 81\ u$, is the current mass-defect, which is effectively a negative value. Therefore, the mass of an atom of helium is less than the mass of the six particles put together. In fact, the helium atom is lighter by about $0{,}031\ 473\ 81\ u$. Some of the mass has gone missing. Where should we find it?

Henceforth, the missing mass has been converted in energy. Therefore, the energy and mass are equivalent. The mass is just a "solid" form of energy. One can try to convert one to the other and back without breaking the law of conservation of matter. Taking into account adjustments introduced before into the relation (1), the physical quantity that mass-defect ΔM represented is not mass-excess (M_A–A) but in reality, it represents molar mass-defect. Because $\Delta m=\Delta M/N_A$ and $m({}^{A}X)=M_A/N_A$, the mass defect associated to an individual nucleus will be

$$\Delta m=m({}^{A}_{Z}X)-m(A,Z) \quad (5)$$

and substituting $m(A,Z)$ by its mathematical expression (4) within relation (5), we obtain the following relation :

$$\Delta m=m({}^{A}_{Z}X)-Z(m_e+m_p)-(A-Z)m_n \quad (6)$$

Now, replacing mass-defect Δm by the relation (6) into Aston's packing fraction (2), we found a new relation which is :

$$f=\frac{\Delta m}{A}=\frac{m({}^{A}_{Z}X)-Z(m_e+m_p)-(A-Z)m_n}{A} \quad (7)$$

where $m({}^{A}_{Z}X)$ is the mass of an atom. The new form of packing fraction is in *u* unit. Indeed, the relation (7) represents *the average mass-defect per nucleon* and probably this is exactly the relationship (in mass unit) that Aston wanted to use to achieve the 1927 Aston's curve (Fig. 2) instead of the (2) relationship.

With definition (6), all stable nuclei are found to have negative Δm values, this is where the sign minus (–) comes from, justifying the use of the term "mass defect". Figure 3 shows the mass Aston's curve for all stable elements on the periodic table. As we can see, mass packing fraction is negative for all existing nuclei except the hydrogen atom for which it is positive.

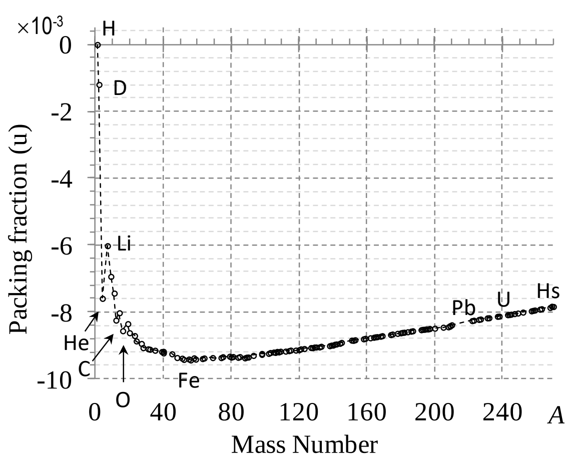

Fig. 3. Mass Aston's curve.

As a result, some mass gets transformed into energy in the formation of nucleus. Thus nuclei having negative value of mass packing fraction are more stable. The greater the negative value of the packing fraction, the greater the loss of mass of its nucleus and hence the greater will be the binding energy (Choppin *et al*., 2001). Except the hydrogen element, positive packing fractions cannot exist because the nucleons into nuclei are bonded (Rideout, 2011). Indeed, when a nucleus receives certain definite quantity of energy (quantified amount of energy) it will be in an excited state and will become unstable (Hecht, 2007); in this case, the packing fraction will increase depending on the size of the amount of energy received, but always keeping the negative sign as long as the nucleus exists. Generally, the lower the packing fraction of an element, the greater the stability of its nucleus.

At this level of development, we need to introduce Einstein's famous law which is $E=\Delta mc^2$. This law allows us to find the relation which exists between average binding energy by nucleon which we denote it ξ_B, mass-defect Δm from relation (6) and the mass Aston's packing fraction f from relation (7) for a given nucleus which is

$$\xi_B=\frac{\Delta mc^2}{A}=fc^2 \quad (8)$$

where c is the speed of the light in the vacuum (c=2,997 924 58 $\times 10^8$ m/s). Because 1 $c^2 \cong$ 931,5 MeV/u, to obtained ξ_B in MeV (Megaelectron-volts), simply multiply f *or* $\Delta m/A$ *wich is* in u by 931,5 value.

As shown on the graph in Figure 4, the average binding energy by nucleon, ξ_B, of nuclei in periodic table are negatives except the hydrogen atom, hence those elements are relatively stables. We note that the value of ξ_B varies in the rather narrow range [-9, 0] MeV.

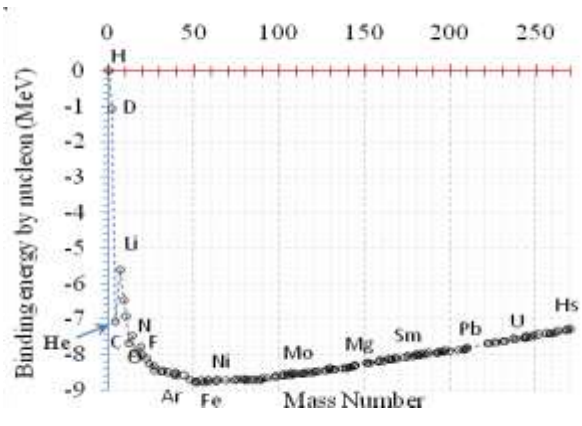

Fig. 4. Binding energy Aston's curve.

Then, to calculate the total energy liberated by the nucleus of an atom, we just need to mu ltiply ξ_B in relation (8) by its mass number A and we obtain

$$E_B=A\xi_\ell \quad (9)$$

Substituting relation (8) by its expression into relation (9) and Δm by its expression (6), and we obtain

$$E_B=\Delta mc^2= [m({}^{A}_{Z}\mathrm{X})-Z(m_e+m_p)-(A-Z)m_n]c^2 \quad (10)$$

A useful practical relation to calculate the binding energy can be deducted from what precedes would be,

$$E_B=931,5\times\Delta m \quad (11)$$

Where the input Δm into this function is in u and the output E_B will be in MeV. This relationship between energy and mass would indicate that in the formation of deuterium by combination of a proton, a neutron and an electron together, the amount of material lost or the mass-defect $-0,002\ 371\ 47\ u$ would be observed as the liberation of an equivalent amount of energy during the formation of this nucleus which is

$$E_B=-0,002\ 371\ 47\times 931,5 = -2,209\ MeV \quad (12)$$

This is quite a small amount of energy in the everyday world, but for a given big quantity of matter such in stars or nuclear reactors and atomic bombs, it will be colossal. But better indication on the stability of a nucleus is obtained when the binding energy is divided by the total number of nucleons into a nucleus to give the average binding energy by nucleon. We can write

$$\xi_\ell = \frac{E_B}{A} \quad (13)$$

It allows comparing the stability of an element with that of another one. For the deuterium, ${}^{2}_{1}\mathrm{H}$, the value of E_B/A for the bond between any two nucleons (Schaeffer, 2012) is equal to – 2,209/2 = – 1,11 MeV, whereas for ${}^{4}_{2}\mathrm{He}$ it is –7,07 MeV. Because – 7,07 < – 1,11; the ${}^{4}_{2}\mathrm{He}$ nucleus is more stable than the ${}^{2}_{1}\mathrm{H}$ nucleus. The lowest values of the average binding energy by nucleon (Fig. 4 or 5) are observed for transition metals (Fe, Co, Ni,...) which indicate maximum stability of their nuclei. Henceforth, on moving to the left or to the right, the values again raise showing increasing instability of nuclei. This is demonstrated by the phenomenon of radioactivity exhibited by high atomic weight elements. The nuclei in such case disintegrate emitting α, β^-, β^+ particles and/or γ radiations giving a lighter and more stable nucleus. From this curve, it is immediately apparent that fusion of light elements like ${}^{1}_{1}\mathrm{H}$ and ${}^{4}_{2}\mathrm{He}$ into heavy ones is highly exothermic, as fission of the heavy elements into lighter atoms, especially present in systems like the sun and stars. Indeed, a few minutes after the Big Bang, the universe contained no other elements than hydrogen and helium (Hinke *et al*., 2012; Haxel *et al*., 1949).

On the plot, we see that certain numbers of nucleons form especially stable nuclei. That effect is observed as small pseudo-periodic valleys which appear spaced out on the x-axis logarithmic scale (Fig. 5). The existence of nuclei with magic numbers (Steppenbeck *et al*., 2013) suggests closed shell configurations, as the orbits of atoms.

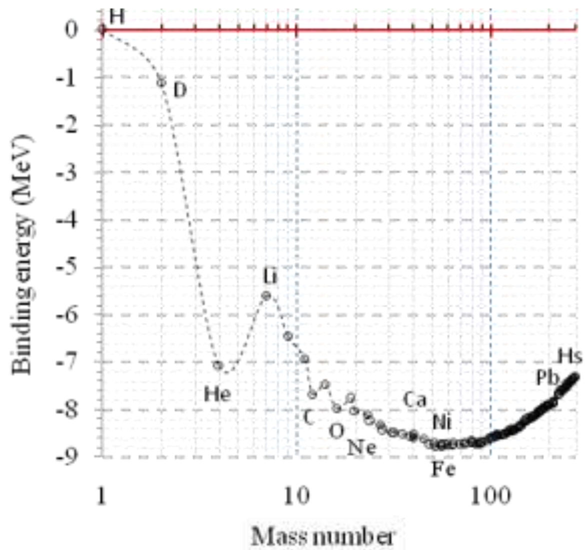

Fig. 5. Some magic elements on the binding energy log-curve.

Now, it is time to introduce a very interesting method to calculate the binding energy step by step, in only four steps as follows.

1st step : writing the nuclear reaction of the nucleus.

$$Z({}^{0}_{-1}e + {}^{1}_{1}p) + (A-Z)\ {}^{1}_{0}n \rightarrow {}^{A}_{Z}\mathrm{X}$$

2nd step: calculating mass-defect.

$$\Delta m = m({}^{A}_{Z}\mathrm{X}) - Z(m_e+m_p) - (A-Z)m_n$$

3rd step : calculating binding energy.

$$E_B = \Delta mc^2 = Afc^2$$

4th step: calculating the average binding energy by nucleon.

$$\xi_B = \frac{E_B}{A}$$

The obtained value of E_B/A should fulfill Aston's condition which is – 9 MeV < ξ_B < 0 MeV.

As example of application, the atomic mass of cobalt-60, ${}^{60}_{27}Co$, is M_A=*59,933 817 17 g/mol*.
What is the average binding energy by nucleon? The conversion factor of mass to energy is 1 u=931,494 028 23 MeV/c^2. By definition, 1 u = 1g/mol.

First step:

$$27({}^{0}_{-1}\mathrm{e} + {}^{1}_{1}\mathrm{p}) + 33\ {}^{1}_{0}\mathrm{n} \rightarrow {}^{60}_{27}\mathrm{C}o$$

Second step : $\Delta m = m({}^{A}_{Z}\mathrm{X}) - m(A,Z)$

$\Delta m = m({}^{A}_{Z}\mathrm{X}) - Z(m_e+m_p) - (A-Z)m_n$

$\Delta m = m({}^{60}_{27}\mathrm{Co}) - 27(m_e+m_p) - 33m_n$

$\Delta m = 59{,}933\ 817\ 17 - 27(0{,}000\ 548\ 58+1{,}007\ 275\ 47) - 33\times 1{,}008\ 664\ 92$

Then,
$\Delta m = -0{,}563\ 348\ 61\ u$

Third step: $E_B=\Delta mc^2$.
$E_B = -0{,}593\ 347\ 61 \times 931{,}494\ 028\ 2$
$E_B = -524{,}754\ 935\ MeV$

Fourth step:

$$\xi_B = \frac{E_B}{A} = -\frac{524{,}754\,9\,935}{60} = -\ 8{,}75 MeV$$

Finally, we find that
-9 MeV < $-8{,}75$ MeV < 0 MeV

CONCLUSION

The calculation of the nuclear binding energy has been exhaustively revised based on the original idea of Aston's whole number. A brief history is presented. The calculation of the nuclear binding energy is done through the famous formula of Albert Einstein on the mass-energy equivalence (E = mc^2). The concepts of Aston's whole number, mass defect and nuclear binding energy are very well defined and a new method for fast and easy calculation of the average nuclear binding energy (energy per nucleon) is proposed. Finally, this work has removed the ambiguity in the sign of the nuclear binding energy which should be therefore negative.

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REFERENCES

Aston, FW. 1920. Isotopes and Atomic Weights. Nature 105:617-619.

Aston, FW. 1919. The mass-spectra of chemical elements Phil. Mag. 38:707-723.

Aston, FW. 1919. The Constitution of the Elements. Nature. 104:393-393.

Aston, FW. 1942. Mass Spectra and Isotopes. (2nd ed.). Edward Arnold & Co., London. pp. 167.

Audi, G. 2006. The history of nuclidic masses and of their evaluation. IJMS. 251:85-94.

Chadwick, J. 1932. The Existence of a Neutron. Proc. R. Soc. London. A 136(830):692-708.

Choppin, G., Liljenzin JO. and Rydberg, J. 2001 Radiochemistry and nuclear chemistry. (3rd edi.). Butterworth-Heinemann. 41-57.

Elsasser, WM. 1933. Sur le principe de Pauli dans les noyaux. J. Phys. Rad. 4:549-556.

Haxel, O., Jensen, JHD. and Suess, HE. 1949. Magic Numbers in Nuclear Structure. Phys. Rev. 75:1766.

Hecht, E. 2007. Physique: Ondes, optique et physique moderne. De Boeck. 365-367.

Hinke CB. *et al*., 2012. Superallowed Gamow–Teller decay of the doubly magic nucleus ^{100}Sn. Nature. 486 (7403):399-405.

Raj, G. 2008. Advanced Inorganic Chemistry. (31st edi.). Krishna Prakashan Media.

Rideout, K. 2011. Binding together bonds. Phys. Teach. 49(3):188-188.

Schaeffer, B. 2012. Ab initio calculation of ${}^{2}\mathrm{H}$ and ${}^{4}\mathrm{He}$.
Binding Energies. J. Modern Physics. 3:1709-1715.

Sharma, BK. 2001. Nuclear and Radiation Chemistry. Krishna Prakashan Media Ltd. 66-67.

Squires, G. 1998. Francis Aston and the mass spectrograph. J. Chem. Soc., Dalton Trans. 23:3893-389.

Steppenbeck, D. *et al*., 2013. Evidence for a new nuclear 'magic number' from the level structure of ^{54}Ca. Nature. 502:207-210.

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